

# On the quadrature methods for the numerical solution of singular integral equations

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## ABSTRACT

A Cauchy type singular integral equation can be numerically solved by the use of an appropriate numerical integration rule and the reduction of this equation to a system of linear algebraic equations, either directly or after the reduction of the Cauchy type singular integral equation to an equivalent Fredholm integral equation of the second kind. In this paper two fundamental theorems on the equivalence (under appropriate conditions) of the aforementioned methods of numerical solution of Cauchy type singular integral equations are proved in sufficiently general cases of Cauchy type singular integral equations of the second kind.

## 1. INTRODUCTION

Several problems of practical interest are reducible to singular integral equations with Cauchy type kernels (called from now on simply singular integral equations). For this reason, several methods for the numerical solution of this class of equations appeared, especially during the last decade [1]. A lot of these methods are based on the use of appropriate numerical integration rules for the approximation of the integrals involved either in the original singular integral equation or in the equivalent Fredholm integral equation of the second kind, obtained by the regularization procedure [2] (especially by solving the dominant equation of the singular integral equation). These methods can be called quadrature methods for the numerical solution of singular integral equations and the first of them, applicable to the original singular integral equation, direct quadrature method. A moderately detailed literature on the quadrature methods of numerical solution of singular integral equations is reported in [3]. In this paper we will consider singular integral equations of the second kind and we will show (under appropriate conditions) the equivalence of the application of quadrature methods to the original singular integral equation and to its equivalent Fredholm integral equations of the second kind. The proof of these results is sufficiently simple in the cases considered, but it seems to be of sufficient importance in further investigations, particularly from the abstract point of view, on the quadrature methods of numerical solution of singular integral equations. Finally, it should be mentioned that the results of section 3 of this paper are a generalization to the case of singular integral equations of the second kind of the relevant results contained in [4-6] and concerning only singular integral equations of the first kind.

## 2. THE FIRST QUADRATURE METHOD

We consider at first the singular integral equation

$$a(x)\varphi(x) + \int_{-1}^1 H(t,x)\varphi(t)dt = f(x), \quad -1 < x < 1, \quad (1)$$

where the kernel  $H(t,x)$  is of the form

$$H(t,x) = \frac{b(x)}{\pi(t-x)} + k(t,x). \quad (2)$$

In these equations the functions  $a(x)$ ,  $b(x)$ ,  $k(t,x)$  and  $f(x)$  are assumed to be known regular functions, whereas  $\varphi(x)$  is the unknown function of the singular integral equation (1). A complete investigation from the theoretical point of view of equation (1) is contained in [2]. Here we assume simply that  $b(x) \neq 0$  and we define the function  $r(x)$  by [1]

$$r^2(x) = a^2(x) + b^2(x). \quad (3)$$

By use of the standard regularization procedure (by solution of the dominant equation of the singular integral equation (1)) [2], equation (1) can be reduced to the following Fredholm integral equation of the second kind [1,2]

$$\varphi(x) + \int_{-1}^1 K(y,x)\varphi(y)dy = F(x), \quad -1 < x < 1, \quad (4)$$

where the new kernel  $K(y,x)$  is determined by

$$K(y,x) = \frac{a(x)}{r^2(x)}k(y,x) - \frac{b(x)Z(x)}{\pi r(x)} \int_{-1}^1 \frac{k(y,t)dt}{r(t)Z(t)(t-x)}. \quad (5)$$

In this equation  $Z(x)$  denotes the fundamental function of the singular integral equation (1) defined in [1,2]. Moreover, the right-hand side function  $F(x)$  of equation (4) is determined by

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$$F(x) = \frac{a(x)}{r^2(x)} f(x) - \frac{b(x)Z(x)}{\pi r(x)} \int_{-1}^1 \frac{f(t)dt}{r(t)Z(t)(t-x)} + \frac{b(x)Z(x)}{r(x)} P_{\kappa-1}(x), \quad (6)$$

where  $P_{\kappa-1}(x)$  is an arbitrary polynomial of degree  $(\kappa-1)$ , where  $\kappa$  is the index of the singular integral equation (1) [1, 2].

The solution  $\varphi(x)$  of equation (1) presents in general singularities near the end-points  $x = \pm 1$  of the integration interval  $[-1, 1]$  behaving like [1, 2]

$$\varphi(x) = w(x)g(x), \quad (7)$$

where  $g(x)$  is a regular function near these points, which will be considered as the new unknown function of equation (1), and  $w(x)$  is a weight function of the form

$$w(x) = (1-x)^a(1+x)^\beta. \quad (8)$$

The exponents  $a$  and  $\beta$  in equation (8) are easily determined from equation (1) [1, 2]. Similarly, the fundamental function  $Z(x)$  of equation (1) behaves near the points  $x = \pm 1$  as

$$Z(x) = w(x)J(x), \quad (9)$$

where  $J(x)$  is a regular function near these points.

The appearance of the weight function  $w(x)$  in the integral terms of equations (1) and (4), because of equation (7), permits us to use a numerical integration rule of the form [7]

$$\int_{-1}^1 w(t)g(t)dt \approx \sum_{i=1}^n A_i g(t_i) \quad (10)$$

for the approximation of the integrals which have  $w(x)$  as a weight function. This rule is in principle applicable only to regular integrals and not to Cauchy type principal value integrals.

We denote by  $g_1(x)$  the approximation of the solution  $g(x)$  (equation (7) taken also into account) of equation (1) when the quadrature rule is used for the approximation of the regular part only of the integral term of this equation. Moreover, we denote by  $g_2(x)$  the approximation of the same solution  $g(x)$  of equation (4) when the same quadrature rule (10) is used for its approximate solution. Then we have

$$a(x)w(x)g_1(x) + \frac{b(x)}{\pi} \int_{-1}^1 w(t) \frac{g_1(t)}{t-x} dt + \sum_{i=1}^n A_i k(t_i, x) g_1(t_i) = f(x), \quad -1 < x < 1, \quad (11)$$

as well as that

$$w(x)g_2(x) + \sum_{i=1}^n A_i K(t_i, x) g_2(t_i) = F(x), \quad -1 < x < 1, \quad (12)$$

respectively.

Equation (11) can be considered once more as a Cauchy type singular integral equation with

$$f^*(x) = f(x) - \sum_{i=1}^n A_i k(t_i, x) g_1(t_i), \quad (13)$$

where the values  $g_1(t_i)$  are assumed for the moment as known constants. Then this equation possesses the closed-form solution [1, 2]

$$w(x)g_1(x) = F^*(x), \quad -1 < x < 1, \quad (14)$$

where, because of equations (6) and (13), we have

$$F^*(x) = \frac{a(x)}{r^2(x)} [f(x) - \sum_{i=1}^n A_i k(t_i, x) g_1(t_i)] - \frac{b(x)Z(x)}{\pi r(x)} \int_{-1}^1 \frac{f(t) - \sum_{i=1}^n A_i k(t_i, t) g_1(t_i)}{r(t)Z(t)(t-x)} dt + \frac{b(x)Z(x)}{r(x)} P_{\kappa-1}(x). \quad (15)$$

By taking again into account equation (6), we can rewrite equation (15) as

$$F^*(x) = F(x) - \sum_{i=1}^n A_i K(t_i, x) g_1(t_i), \quad (16)$$

where equation (5) was also taken into account.

By substituting this value of  $F^*(x)$  into equation (14), we obtain

$$w(x)g_1(x) + \sum_{i=1}^n A_i K(t_i, x) g_1(t_i) = F(x), \quad -1 < x < 1. \quad (17)$$

Finally, by comparing equations (12) and (17), we conclude that

$$g_1(x) \equiv g_2(x). \quad (18)$$

Of course, the determination of  $g_{1,2}(x)$  from equation (12) (or equation (17)) can be made by the well-known Nyström (quadrature) method [8]. In accordance with this method, equation (12) is applied at the nodes  $t_i$  of the quadrature rule (10). Then a system of  $n$  linear algebraic equations is obtained for the determination of the values of  $g_{1,2}(t_i)$ . After the determination of these values,  $g_{1,2}(x)$  are determined by using the natural interpolation formula of Nyström [8], which coincides with equation (14) with  $F^*(x)$  determined from equation (16).

Equation (18) constitutes a fundamental result for the quadrature methods of numerical solution of singular integral equations, in spite of the fact that it may be considered trivial or obvious. This result can be stated in the form of the following theorem:

#### Theorem 1

Under the assumptions of this section, the approximate solution of a singular integral equation of the first or the second kind by reduction to an equivalent Fredholm integral equation of the second kind and approximate solution of the latter by the Nyström (quadrature) method (based on an appropriate numerical integration rule), coincides with the approximate solution of the original singular integral equation after the same numerical integration rule is used for the approximation of only the regular part of the integral term.

Clearly, the above theorem holds true independently of

the quadrature rule (10) used. Moreover, it holds true for any finite integration interval  $[c, d]$  and not only for the interval  $[-1, 1]$ . Of course, the problems of convergence of the quadrature rule (10) and the good or bad approximation of  $g(x)$  by  $g_{1,2}(x)$  are of no interest for the results of this section.

### 3. THE SECOND QUADRATURE METHOD

We come now to the direct numerical solution of equation (1) by the quadrature method. The literature on this method is reported in [3, 9-11]. The relation between the approximate solution  $g_3(x)$  of a singular integral equation by the direct quadrature method and the corresponding solution  $g_4(x)$  of the equivalent Fredholm integral equation of the second kind was considered in [4-6], but only for singular integral equations of the first kind. Here we will consider singular integral equations of the second kind of the form (1), but with the functions  $a(x)$  and  $b(x)$  assumed to be reduced to constants,  $a$  and  $b$ , respectively, along the integration interval  $[-1, 1]$ . Moreover, we will use the Gauss-Jacobi quadrature rule [7], which is of the form (10) for regular integrals. The results of this section seem not to hold when  $a(x)$  and  $b(x)$  in equation (1) are not constants or an arbitrary quadrature rule of the form (10) is used. The Gauss-Jacobi quadrature rule was applied to the evaluation of Cauchy type principal value integrals for the first time in [12]. Here we use it in a somewhat modified form taking into account the dominant part of our singular integral equation

$$aw(x)g(x) + \frac{b}{\pi} \int_{-1}^1 w(t) \frac{g(t)}{t-x} dt + \int_{-1}^1 w(t)k(t,x)g(t)dt = f(x), \quad -1 < x < 1, \quad (19)$$

resulting from equation (1) under the present assumptions, as well as equations (2) and (7). In our case, the Gauss-Jacobi quadrature rule takes the form [13]

$$aw(x)g(x) + \frac{b}{\pi} \int_{-1}^1 w(t) \frac{g(t)}{t-x} dt \approx \frac{b}{\pi} \sum_{i=1}^n A_i \frac{g(t_i)}{t_i-x} - \frac{b}{2\sin\pi a} \frac{P_{n-1}^{(-a,-\beta)}(x)}{P_n^{(a,\beta)}(x)} g(x), \quad x \neq t_i, \quad i = 1(1)n, \quad (20)$$

where  $P_n^{(a,\beta)}(x)$  denotes the Jacobi polynomial of degree  $n$  associated with the constants  $a$  and  $\beta$  and the index  $\kappa = -(a+\beta)$  [1, 2] of equation (19) is assumed equal to 1. The results of this section hold also true even when  $\kappa=0$ . Moreover, since  $\kappa=1$ , equation (19) should be supplemented by an additional condition, for example

$$\int_{-1}^1 w(t)g(t)dt = 0. \quad (21)$$

Now, by applying equations (10) and (20) to equation (19), we obtain

$$\sum_{i=1}^n A_i \left[ \frac{b}{\pi(t_i-x)} + k(t_i, x) \right] g_3(t_i) - \frac{b}{2\sin\pi a} \frac{P_{n-1}^{(-a,-\beta)}(x)}{P_n^{(a,\beta)}(x)} g_3(x) = f(x), \quad x \neq t_i, \quad i = 1(1)n, \quad (22)$$

for the approximation  $g_3(x)$  of the solution  $g(x)$  of equation (19). This equation can also be used as a natural interpolation formula [6] for  $g_3(x)$ , that is

$$g_3(x) = \frac{2}{b} \sin\pi a \frac{P_n^{(a,\beta)}(x)}{P_{n-1}^{(-a,-\beta)}(x)} \left[ \sum_{i=1}^n A_i \left[ \frac{b}{\pi(t_i-x)} + k(t_i, x) \right] g_3(t_i) - f(x) \right], \quad x \neq t_i, \quad i = 1(1)n, \quad x \neq x_k, \quad k = 1(1)(n-1), \quad (23)$$

where  $x_k$  are the roots of  $P_{n-1}^{(-a,-\beta)}(x)$ , that is

$$P_{n-1}^{(-a,-\beta)}(x_k) = 0, \quad k = 1(1)(n-1). \quad (24)$$

For the numerical evaluation of the values of  $g_3(t_i)$ , we apply equation (22) at the collocation points  $x_k$  [14, 15], that is

$$\sum_{i=1}^n A_i \left[ \frac{b}{\pi(t_i-x_k)} + k(t_i, x_k) \right] g_3(t_i) = f(x_k), \quad k = 1(1)(n-1). \quad (25)$$

Moreover, we apply the quadrature rule (10) to equation (21) and we find

$$\sum_{i=1}^n A_i g_3(t_i) = 0. \quad (26)$$

From the system of linear algebraic equations (25) and (26), we determine the values of  $g_3(t_i)$ . Then equation (23) is the natural interpolation formula [6] for the determination of  $g_3(x)$  along the interval  $[-1, 1]$ .

Evidently, the values of  $g_3(t_i)$  are different from the values of  $g_{1,2}(t_i)$  obtained by the method reported in the previous section, since here not only the regular but also the Cauchy type part of the integral term of our singular integral equation has been approximated by using a quadrature rule. Here we will consider again the Fredholm integral equation of the second kind which is equivalent to the singular integral equation (19). This equation has the form [1, 2, 14]

$$(a^2 + b^2)g(x) + \int_{-1}^1 w(y)K(y,x)g(y)dy = F(x), \quad -1 < x < 1, \quad \text{where now} \quad (27)$$

$$K(y, x) = aw^*(x)k(y, x) - \frac{b}{\pi} \int_{-1}^1 w^*(t) \frac{k(y, t)}{t-x} dt \quad (28)$$

and

$$F(x) = aw^*(x)f(x) - \frac{b}{\pi} \int_{-1}^1 w^*(t) \frac{f(t)}{t-x} dt, \quad (29)$$

with

$$w^*(x) = (1-x)^{-a}(1+x)^{-\beta}. \quad (30)$$

For the numerical solution of equation (27) by the Nyström (quadrature) method, we will apply the quadrature rule (10) to its integral term as we made in the previous section. But here we will apply also the quadrature rule

$$aw^*(x)g(x) - \frac{b}{\pi} \int_{-1}^1 w^*(t) \frac{g(t)}{t-x} dt \approx -\frac{b}{\pi} \sum_{k=1}^{n-1} B_k \frac{g(x_k)}{x_k-x} - \frac{2b}{\sin \pi a} \frac{P_n^{(a, \beta)}(x)}{P_n^{(-a, -\beta)}(x)} g(x), \quad (31)$$

analogous to the quadrature rule (20), but for the weight function  $w^*(x)$ , for the approximation of the integrals in equations (28) and (29). Then we obtain from equation (27)

$$(a^2+b^2)g_4(x) + \sum_{i=1}^n A_i \left[ -\frac{b}{\pi} \sum_{k=1}^{n-1} B_k \frac{k(t_i, x_k)}{x_k-x} - \frac{2b}{\sin \pi a} \frac{P_n^{(a, \beta)}(x)}{P_n^{(-a, -\beta)}(x)} k(t_i, x) \right] g_4(t_i) = -\frac{b}{\pi} \sum_{k=1}^{n-1} B_k \frac{f(x_k)}{x_k-x} - \frac{2b}{\sin \pi a} \frac{P_n^{(a, \beta)}(x)}{P_n^{(-a, -\beta)}(x)} f(x), \quad -1 < x < 1, x \neq x_k, k = 1(1)(n-1). \quad (32)$$

By using the nodes  $t_i$  as collocation points, we obtain from this equation the following system of  $n$  linear algebraic equations for the determination of the values  $g_4(t_i)$  of the approximation  $g_4(x)$  of the unknown function  $g(x)$  of equation (27) at the nodes  $t_i$

$$(a^2+b^2)g_4(t_\ell) + \frac{b}{\pi} \sum_{i=1}^n A_i \left[ \sum_{k=1}^{n-1} B_k \frac{k(t_i, x_k)}{t_\ell-x_k} \right] g_4(t_i) = \frac{b}{\pi} \sum_{k=1}^{n-1} B_k \frac{f(x_k)}{t_\ell-x_k}, \ell = 1(1)n. \quad (33)$$

To obtain these equations, we have taken into account that the nodes  $t_i$  in equations (10) and (20) are the roots of the Jacobi polynomial  $P_n^{(a, \beta)}(x)$ , that is

$$P_n^{(a, \beta)}(t_i) = 0, i = 1(1)n. \quad (34)$$

After the determination of the values of  $g_4(t_i)$  from equations (33),  $g_4(x)$  can be determined along almost

the whole integration interval  $[-1, 1]$  from equation (32) as follows

$$g_4(x) = \frac{1}{a^2+b^2} \left[ \frac{b}{\pi} \sum_{k=1}^{n-1} \frac{B_k}{x_k-x} \left[ \sum_{i=1}^n A_i k(t_i, x_k) g_4(t_i) - f(x_k) \right] + \frac{2b}{\sin \pi a} \frac{P_n^{(a, \beta)}(x)}{P_n^{(-a, -\beta)}(x)} \left[ \sum_{i=1}^n A_i k(t_i, x) g_4(t_i) - f(x) \right] \right], \quad x \neq x_k, k = 1(1)(n-1). \quad (35)$$

We will show now that equations (23) and (35) coincide, that is

$$g_3(x) \equiv g_4(x). \quad (36)$$

This will be made by using a procedure similar to that used in [4-6] for singular integral equations of the first kind.

To this end, we will show at first that the systems of linear algebraic equations (25, 26) and (33) give the same numerical results, that is

$$g_3(t_i) = g_4(t_i), i = 1(1)n. \quad (37)$$

By multiplying each one of equations (25) by  $(b/\pi)B_k/(t_\ell-x_k)$  and equation (26) by  $(b/\pi)\epsilon$ , where  $\epsilon$  is an arbitrary constant, we obtain from equations (25, 26) the following system of linear algebraic equations

$$\frac{b}{\pi} \sum_{i=1}^n A_i \left[ \frac{b}{\pi} \sum_{k=1}^{n-1} \frac{B_k}{(t_i-x_k)(t_\ell-x_k)} + \epsilon \right] g_3(t_i) + \frac{b}{\pi} \sum_{i=1}^n A_i \left[ \sum_{k=1}^{n-1} B_k \frac{k(t_i, x_k)}{t_\ell-x_k} \right] g_3(t_i) = \frac{b}{\pi} \sum_{k=1}^{n-1} B_k \frac{f(x_k)}{t_\ell-x_k}, \ell = 1(1)n. \quad (38)$$

By comparing the sets of equations (33) and (38), we see that they coincide provided that the following equations hold true

$$\frac{b}{\pi} A_i \left[ \frac{b}{\pi} \sum_{k=1}^{n-1} \frac{B_k}{(t_i-x_k)(t_\ell-x_k)} + \epsilon \right] = (a^2+b^2)\delta_{i\ell}, \quad i, \ell = 1(1)n, \quad (39)$$

(where  $\delta_{i\ell}$  denotes the Kronecker delta) for an appropriate selection of the constant  $\epsilon$ . By taking into account the well-known identity [14]

$$aw(x)P_k^{(a, \beta)}(x) + \frac{b}{\pi} \int_{-1}^1 w(t) \frac{P_k^{(a, \beta)}(t)}{t-x} dt = -2^{-\kappa} \frac{b}{\sin \pi a} P_{k-\kappa}^{(-a, -\beta)}(x), -\kappa = a+\beta, -1 < x < 1, \quad (40)$$

and applying it for the weight function  $w^*(x)$ , defined by equation (30), and for  $k=0$ , we obtain

$$-aw^*(x) + \frac{b}{\pi} \int_{-1}^1 \frac{w^*(t)}{t-x} dt = \frac{2b}{\sin \pi a} P_1^{(a, \beta)}(x), \quad (41)$$

where we have taken into account that [16]

$$P_0^{(a,\beta)}(x) \equiv 1, \quad (42)$$

as well as the fact that we have assumed that the index of equation (19) is equal to 1, that is

$$\kappa = -(a+\beta) \equiv 1. \quad (43)$$

Similarly, we apply the quadrature rule (31) for  $g(x) \equiv 1$  (then this rule is exact) and we find

$$\begin{aligned} & -aw^*(x) + \frac{b}{\pi} \int_{-1}^1 \frac{w^*(t)}{t-x} dt \\ &= \frac{b}{\pi} \sum_{k=1}^{n-1} \frac{B_k}{x_k - x} + \frac{2b}{\sin \pi a} \frac{P_n^{(a,\beta)}(x)}{P_{n-1}^{(-a,-\beta)}(x)}. \end{aligned} \quad (44)$$

By comparing equations (41) and (43) and applying the resulting equation at the nodes  $t_i$  of the quadrature rules (10) and (20) (determined from equations (34)), we find

$$\frac{b}{\pi} \sum_{k=1}^{n-1} \frac{B_k}{x_k - t_i} = \frac{2b}{\sin \pi a} P_1^{(a,\beta)}(t_i), \quad i = 1(1)n. \quad (45)$$

But we have also that

$$\begin{aligned} & \frac{b}{\pi} \sum_{k=1}^{n-1} \frac{B_k}{(t_i - x_k)(t_\ell - x_k)} \\ &= \frac{b}{\pi(t_i - t_\ell)} \sum_{k=1}^{n-1} B_k \left( \frac{1}{x_k - t_i} - \frac{1}{x_k - t_\ell} \right), \quad i, \ell = 1(1)n. \end{aligned} \quad (46)$$

For  $i \neq \ell$  we obtain further (taking also into account equations (45))

$$\begin{aligned} & \frac{b}{\pi} \sum_{k=1}^{n-1} \frac{B_k}{(t_i - x_k)(t_\ell - x_k)} \\ &= \frac{1}{t_i - t_\ell} \frac{2b}{\sin \pi a} \left[ P_1^{(a,\beta)}(t_i) - P_1^{(a,\beta)}(t_\ell) \right], \quad i \neq \ell, \quad i, \ell = 1(1)n. \end{aligned} \quad (47)$$

But since

$$P_1^{(a,\beta)}(x) = \frac{1}{2}(a+\beta+2)x + \frac{1}{2}(a-\beta), \quad (48)$$

as is well-known for Jacobi polynomials [16], we can see from equations (47) that equations (39) hold true for  $i \neq \ell$  provided that

$$\epsilon = -b(a+\beta+2)/\sin \pi a. \quad (49)$$

To complete the proof of (39), we have to show that

$$\begin{aligned} & \frac{b}{\pi} A_i \left[ \frac{b}{\pi} \sum_{k=1}^{n-1} \frac{B_k}{(x_k - t_i)^2} - \frac{b(a+\beta+2)}{\sin \pi a} \right] = a^2 + b^2, \\ & i = 1(1)n. \end{aligned} \quad (50)$$

To do so, we take into account equations (41), (44) and (48) and we find

$$\begin{aligned} & \frac{b}{\pi} \sum_{k=1}^{n-1} \frac{B_k}{x_k - x} + \frac{2b}{\sin \pi a} \frac{P_n^{(a,\beta)}(x)}{P_{n-1}^{(-a,-\beta)}(x)} \\ &= \frac{b}{\sin \pi a} [(a+\beta+2)x + a - \beta]. \end{aligned} \quad (51)$$

By differentiating this equation with respect to  $x$  and applying the resulting equation at the nodes  $t_i$  (determined from equations (34)), we find

$$\begin{aligned} & \frac{b}{\pi} \sum_{k=1}^{n-1} \frac{B_k}{(x_k - t_i)^2} + \frac{2b}{\sin \pi a} \frac{P_n^{(a,\beta)'}(t_i)}{P_{n-1}^{(-a,-\beta)}(t_i)} \\ &= \frac{b(a+\beta+2)}{\sin \pi a}, \quad i = 1(1)n. \end{aligned} \quad (52)$$

But on the basis of the results of [14], it can easily be seen that

$$A_i = -\frac{\pi}{2 \sin \pi a} \frac{P_{n-1}^{(-a,-\beta)}(t_i)}{P_n^{(a,\beta)}(t_i)}, \quad i = 1(1)n. \quad (53)$$

Then equation (52) takes the form

$$\begin{aligned} & \frac{b}{\pi} \sum_{k=1}^{n-1} \frac{B_k}{(x_k - t_i)^2} - \frac{b(a+\beta+2)}{\sin \pi a} = \frac{\pi b}{\sin^2 \pi a} \frac{1}{A_i}, \\ & i = 1(1)n. \end{aligned} \quad (54)$$

Hence, to show the validity of equations (50), we have to show that

$$\sin^2 \pi a = b^2/(a^2 + b^2). \quad (55)$$

But this is valid since [14]

$$\tan \pi a = -b/a, \quad \tan \pi b = b/a. \quad (56)$$

Hence, the proof of equations (39) was completed.

We have shown above that the solution  $g_3(t_i)$  ( $i=1(1)n$ ) of equations (25, 26) satisfies also equations (33). It is also possible to show in a similar way that the solution  $g_4(t_i)$  ( $i=1(1)n$ ) of equations (33) satisfies also equations (25, 26). Hence, these two sets of linear algebraic equations are equivalent and equations (37) hold true. Now, on the basis of equations (37), we will show the validity of the identity (36).

By taking into account the natural interpolation formulas (23) and (35) for  $g_3(x)$  and  $g_4(x)$ , respectively, as well as equations (37) and (55), we see that we have to show that

$$\begin{aligned} & \frac{2}{b} \sin \pi a \frac{P_n^{(a,\beta)}(x)}{P_{n-1}^{(-a,-\beta)}(x)} \sum_{i=1}^n A_i \frac{g_3(t_i)}{t_i - x} \\ &= \frac{1}{a^2 + b^2} \sum_{k=1}^{n-1} \frac{B_k}{x_k - x} \left[ \sum_{i=1}^n A_i k(t_i, x_k) g_3(t_i) - f(x_k) \right], \\ & x \neq t_i, \quad i = 1(1)n, \quad x \neq x_k, \quad k = 1(1)(n-1). \end{aligned} \quad (57)$$

By taking further into consideration equations (25, 26), we see that equation (57) can be written as

$$\begin{aligned} & \frac{2}{b} \sin \pi a \frac{P_n^{(a,\beta)}(x)}{P_{n-1}^{(-a,-\beta)}(x)} \sum_{i=1}^n A_i \frac{g_3(t_i)}{t_i - x} \\ &= \frac{b}{\pi(a^2 + b^2)} \sum_{i=1}^n A_i \left[ \sum_{k=1}^{n-1} \frac{B_k}{(x_k - t_i)(x_k - x)} + \epsilon \right] g_3(t_i), \end{aligned}$$

$$x \neq t_i, i = 1(1)n, x \neq x_k, k = 1(1)(n-1), \quad (58)$$

where  $\epsilon$  is given by equation (49). Hence, it should be proved that

$$\begin{aligned} & \frac{2}{b} \sin \pi a \frac{P_n^{(a, \beta)}(x)}{P_{n-1}^{(-a, -\beta)}(x)} \\ &= \frac{b}{\pi(a^2 + b^2)} \left[ \sum_{k=1}^{n-1} B_k \left( \frac{1}{x_k - t_i} - \frac{1}{x_k - x} \right) + \epsilon(t_i - x) \right], \\ & x \neq x_k, k = 1(1)(n-1), \end{aligned} \quad (59)$$

where equation (46) was also taken into account. Now, if we use equation (51) as it stands, as well as after applying it at the nodes  $t_i$  (determined from equations (34)), we find from equations (59)

$$\begin{aligned} & \frac{2}{b} \sin \pi a \frac{P_n^{(a, \beta)}(x)}{P_{n-1}^{(-a, -\beta)}(x)} \\ &= \frac{1}{a^2 + b^2} \left[ \frac{2b}{\sin \pi a} \frac{P_n^{(a, \beta)}(x)}{P_{n-1}^{(-a, -\beta)}(x)} + \frac{b(a + \beta + 2)}{\sin \pi a} (t_i - x) \right. \\ & \left. + \epsilon(t_i - x) \right], x \neq x_k, k = 1(1)(n-1). \end{aligned} \quad (60)$$

The validity of this equation follows directly if equations (49) and (55) are taken into account. This completes the proof of the identity (36).

Similar results can also be proved in the case when the index  $\kappa$  of the original singular integral equation (19) is equal to 0 and not equal to 1 as supposed previously. This is a simpler (but less general) case since no condition of the form (21) is valid. The above results can be stated formally in the form of the following theorem:

#### Theorem 2

Under the assumptions of this section, the approximate solution of a singular integral equation of the second kind by the direct quadrature method, based on the Gauss-Jacobi numerical integration rule with  $n$  nodes, coincides with the approximate solution of the Fredholm integral equation of the second kind equivalent to the original singular integral equation (by using the Nyström (quadrature) method with the same numerical integration rule), provided that the integrals in the kernel and the right-hand side function of the Fredholm integral equation are also approximated by the use of the Gauss-Jacobi quadrature rule for their weight function and with  $(n - \kappa)$  nodes, where  $\kappa$  is the index of the original singular integral equation. Theorems 1 and 2 establish in a satisfactory way the relations between the quadrature methods for the numerical solution of singular integral equations and the corresponding Fredholm integral equations of the second kind. It is believed that these theorems will be proved useful in further investigations of these methods.

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